

p -ADIC MULTIREOLUTION ANALYSIS AND WAVELET FRAMES

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ABSTRACT. We study p -adic multiresolution analyses (MRAs). A complete characterisation of test functions generating MRAs (scaling functions) is given. We prove that only 1-periodic test functions may be taken as orthogonal scaling functions. We also suggest a method for the construction of wavelet functions and prove that any wavelet function generates a p -adic wavelet frame.

1. INTRODUCTION

In the early nineties a general scheme for the construction of wavelets (of real argument) was developed. This scheme is based on the notion of multiresolution analysis (MRA in the sequel) introduced by Y. Meyer and S. Mallat [1], [2] (see also, e.g., [4], [10]). Immediately specialists started to implement new wavelet systems. Nowadays it is difficult to find an engineering area where wavelets are not applied.

In the p -adic setting, the situation is as follows. In 2002 S. V. Kozyrev [3] found a compactly supported p -adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. It turned out that these wavelets were eigenfunctions of p -adic pseudo-differential operators [5]. J.J. Benedetto and R.L. Benedetto [6] conjectured that other p -adic wavelets with the same set of translations can not be constructed because this set is not a group, and the corresponding MRA-theory can not be developed. Another conjecture was raised by A. Khrennikov and V. Shelkovich [7]. They assumed that the equality

$$(1.1) \quad \phi(x) = \sum_{r=0}^{p-1} \phi\left(\frac{1}{p}x - \frac{r}{p}\right), \quad x \in \mathbb{Q}_p,$$

may be considered as a *refinement equation* for the Haar MRA generating Kozyrev's wavelets. A solution ϕ to this equation (*a refinable function*) is the characteristic function of the unit disc. We note that equation (1.1) reflects a *natural* "self-similarity" of the space \mathbb{Q}_p : the unit disc $B_0(0) = \{x : |x|_p \leq 1\}$ is represented as the union $\bigcup_{r=0}^{p-1} B_{-1}(r)$ of p mutually *disjoint* discs $B_{-1}(r) = \{x : |x - r|_p \leq p^{-1}\}$ (see [12, I.3, Examples 1,2.]). Following this idea, the notion of p -adic MRA was introduced and a general scheme for its construction was described in [8]. Also, using (1.1) as a generating refinement equation, this scheme was realized to construct the 2-adic Haar MRA. In contrast to the real setting, the *refinable function* ϕ generating the Haar MRA is *periodic*, which implies the existence of *infinitely many*

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different orthonormal wavelet bases in the same Haar MRA. One of them coincides with Kozyrev's wavelet basis. The authors of [9] described a wide class of functions generating a MRA, but all of these functions are 1-periodic. In the present paper we prove that there exist no other orthogonal test scaling functions generating a MRA, except for those described in [8]. Also, the MRAs generated by arbitrary test scaling functions (not necessary orthogonal) are considered. We thoroughly study these scaling functions and develop a method to construct a wavelet frame based on a given MRA.

Here and in what follows, we shall systematically use the notation and the results from [12]. Let \mathbb{N} , \mathbb{Z} , \mathbb{R} , \mathbb{C} be the sets of positive integers, integers, real numbers, complex numbers, respectively. The field \mathbb{Q}_p of p -adic numbers is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the non-Archimedean p -adic norm $|\cdot|_p$. This p -adic norm is defined as follows: $|0|_p = 0$; if $x \neq 0$, $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers m , n are not divisible by p , then $|x|_p = p^{-\gamma}$. The norm $|\cdot|_p$ satisfies the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$. The canonical form of any p -adic number $x \neq 0$ is

$$(1.2) \quad x = p^\gamma(x_0 + x_1p + x_2p^2 + \cdots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_j \in D_p := \{0, 1, \dots, p-1\}$, $x_0 \neq 0$, $j = 0, 1, \dots$. We shall write the p -adic numbers $k = k_0 + k_1p + \cdots + k_{s-1}p^{s-1}$, $k_j \in D_p$, $j = 0, 1, \dots, s-1$, following the usual form, as in the real analysis: $k = 0, 1, \dots, p^s - 1$.

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq p^\gamma\}$ the disc of radius p^γ with the center at a point $a \in \mathbb{Q}_p$, $\gamma \in \mathbb{Z}$. Any two balls in \mathbb{Q}_p either are disjoint or one contains the other.

There exists the Haar measure dx on \mathbb{Q}_p which is positive, invariant under the shifts, i.e., $d(x+a) = dx$, and normalized by $\int_{|\xi|_p \leq 1} dx = 1$. A complex-valued function f defined on \mathbb{Q}_p is called *locally-constant* if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that $f(x+y) = f(x)$, $y \in B_{l(x)}(0)$. Denote by \mathcal{D} the linear space of locally-constant compactly supported functions (so-called test functions) [12, VI.1.,2.]. The space \mathcal{D} is an analog of the Schwartz space in the real analysis.

The Fourier transform of $\varphi \in \mathcal{D}$ is defined as

$$\widehat{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi \cdot x) \varphi(x) dx, \quad \xi \in \mathbb{Q}_p,$$

where $\chi_p(\xi \cdot x) = e^{2\pi i \{\xi x\}_p}$ is the additive character for the field \mathbb{Q}_p , $\{\cdot\}_p$ is a fractional part of a number $x \in \mathbb{Q}_p$. The Fourier transform is a linear isomorphism taking \mathcal{D} into \mathcal{D} . The Fourier transform is extended to $L^2(\mathbb{Q}_p)$ in a standard way. If $f \in L^2(\mathbb{Q}_p)$, $0 \neq a \in \mathbb{Q}_p$, $b \in \mathbb{Q}_p$, then [12, VII,(3.3)]:

$$(1.3) \quad F[f(ax+b)](\xi) = |a|_p^{-1} \chi_p\left(-\frac{b}{a}\xi\right) F[f(x)]\left(\frac{\xi}{a}\right).$$

According to [12, IV,(3.1)],

$$(1.4) \quad F[\Omega(p^{-k}|\cdot|_p)](x) = p^k \Omega(p^k |x|_p), \quad k \in \mathbb{Z}, \quad x \in \mathbb{Q}_p,$$

where $\Omega(t) = 1$ for $t \in [0, 1]$; $\Omega(t) = 0$ for $t \notin [0, 1]$.

2. MULTIREOLUTION ANALYSIS

Let us consider the set

$$I_p = \{a = p^{-\gamma}(a_0 + a_1p + \cdots + a_{\gamma-1}p^{\gamma-1}) : \gamma \in \mathbb{N}; a_j \in D_p; j = 0, 1, \dots, \gamma-1\}.$$

It is well known that $\mathbb{Q}_p = B_0(0) \cup \bigcup_{\gamma=1}^{\infty} S_\gamma$, where $S_\gamma = \{x \in \mathbb{Q}_p : |x|_p = p^{-\gamma}\}$. Due to (1.2), $x \in S_\gamma$, $\gamma \geq 1$, if and only if $x = x_{-\gamma}p^{-\gamma} + x_{-\gamma+1}p^{-\gamma+1} + \cdots + x_{-1}p^{-1} + \xi$, where $x_{-\gamma} \neq 0$, $\xi \in B_0(0)$. Since $x_{-\gamma}p^{-\gamma} + x_{-\gamma+1}p^{-\gamma+1} + \cdots + x_{-1}p^{-1} \in I_p$, we have a “natural” decomposition of \mathbb{Q}_p into a union of mutually disjoint discs: $\mathbb{Q}_p = \bigcup_{a \in I_p} B_0(a)$. So, I_p is a “natural” set of shifts for \mathbb{Q}_p .

Definition 2.1. A collection of closed spaces $V_j \subset L^2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis (MRA) in $L^2(\mathbb{Q}_p)$* if the following axioms hold

- (a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{Q}_p)$;
- (c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (d) $f(\cdot) \in V_j \iff f(p^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (e) there exists a function $\phi \in V_0$ such that $V_0 := \overline{\text{span}\{\phi(\cdot - a), a \in I_p\}}$.

The function ϕ from axiom (e) is called *scaling*. One also says that a MRA is generated by its scaling function ϕ (or ϕ generates the MRA). It follows immediately from axioms (d) and (e) that

$$(2.1) \quad V_j := \overline{\text{span}\{\phi(p^{-j}x - a), a \in I_p\}}, \quad j \in \mathbb{Z}.$$

An important class of MRAs consists of those generated by so-called *orthogonal scaling functions*. A scaling function ϕ is said to be orthogonal if $\{\phi(\cdot - a), a \in I_p\}$ is an orthonormal basis for V_0 . Consider such a MRA. Evidently, the functions $p^{j/2}\phi(p^{-j}\cdot - a)$, $a \in I_p$, form an orthonormal basis for V_j , $j \in \mathbb{Z}$. According to the standard scheme (see, e.g., [10, §1.3]) for the construction of MRA-based wavelets, for each j , we define a space W_j (*wavelet space*) as the orthogonal complement of V_j in V_{j+1} , i.e., $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$(2.2) \quad f \in W_j \iff f(p^{-1}\cdot) \in W_{j+1}, \quad \text{for all } j \in \mathbb{Z}$$

and $W_j \perp W_k$, $j \neq k$. Taking into account axioms (b) and (c), we obtain

$$(2.3) \quad \bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{Q}_p) \quad (\text{orthogonal direct sum}).$$

If we now find functions $\psi^{(\nu)} \in W_0$, $\nu \in A$, such that the functions $\psi^{(\nu)}(x - a)$, $a \in I_p$, $\nu \in A$, form an orthonormal basis for W_0 , then, due to (2.2) and (2.3), the system $\{p^{j/2}\psi^{(\nu)}(p^{-j}\cdot - a), a \in I_p, j \in \mathbb{Z}, \nu \in A\}$ is an orthonormal basis for $L^2(\mathbb{Q}_p)$. Such a function ψ is called a *wavelet function* and the basis is a *wavelet basis*.

Another interesting class of scaling functions consists of functions ϕ so that $\{\phi(\cdot - a), a \in I_p\}$ is a Riesz system. Probably, adopting the ideas developed for the real setting, one can use MRAs generated by such functions ϕ for construction of dual biorthogonal wavelet systems. This topic is, however, out of our consideration in the present paper.

In Section 3 we will discuss how to construct a p -adic wavelet frame based on an arbitrary MRA generated by a test function.

Let ϕ be an orthogonal scaling function for a MRA $\{V_j\}_{j \in \mathbb{Z}}$. Since the system $\{p^{1/2}\phi(p^{-1} \cdot -a), a \in I_p\}$ is a basis for V_1 in this case, it follows from axiom (a) that

$$(2.4) \quad \phi = \sum_{a \in I_p} \alpha_a \phi(p^{-1} \cdot -a), \quad \alpha_a \in \mathbb{C}.$$

We see that the function ϕ is a solution of a special kind of functional equation. Such equations are called *refinement equations*, and their solutions are called *refinable functions*¹. It will be shown in Section 3 that any test scaling function (not necessary orthogonal) is refinable.

A natural way for the construction of a MRA (see, e.g., [10, §1.2]) is the following. We start with a refinable function ϕ and define the spaces V_j by (2.1). It is clear that axioms (d) and (e) of Definition 2.1 are fulfilled. Of course, not any such function ϕ provides axiom (a). In the real setting, the relation $V_0 \subset V_1$ holds if and only if the refinable function satisfies a refinement equation. The situation is different in the p -adic case. Generally speaking, a refinement equation (2.4) does not imply the including property $V_0 \subset V_1$ because the set of shifts I_p does not form a group. Indeed, we need all the functions $\phi(\cdot - b)$, $b \in I_p$, to belong to the space V_1 , i.e., the identities $\phi(x - b) = \sum_{a \in I_p} \alpha_{a,b} \phi(p^{-1}x - a)$ should be fulfilled for all $b \in I_p$. Since $p^{-1}b + a$ is not in I_p in general, we can not state that $\phi(x - b)$ belongs to V_1 for all $b \in I_p$. Nevertheless, we will see below that a wide class of refinable equations provide the including property.

Providing axiom (a) is a key moment for the construction of MRA. Axioms (b) and (c) are fulfilled for a wide class of functions ϕ because of the following statements.

Theorem 2.2. *If $\phi \in L^2(\mathbb{Q}_p)$ and $\hat{\phi}$ is compactly supported, then axiom (c) of Definition 2.1 holds for the spaces V_j defined by (2.1).*

Proof. Let $\hat{\phi} \subset B_M(0)$, $M \in \mathbb{Z}$. Assume that a function $f \in L^2(\mathbb{Q}_p)$ belongs to any space V_j , $j \in \mathbb{Z}$. Given $j \in \mathbb{N}$ and $\epsilon > 0$, there exists a function $f_\epsilon := \sum_{a \in I_p} \alpha_a \phi(p^j \cdot -a)$, where the sum is finite, such that $\|f - f_\epsilon\| < \epsilon$. Using (1.3), it is not difficult to see that $\text{supp } \hat{f}_\epsilon \subset \text{supp } \hat{\phi}(p^{-j} \cdot)$, which yields that $\hat{f}_\epsilon(\xi) = 0$ for any $\xi \notin B_{M-j}(0)$. Due to the Plancherel theorem, it follows that $\hat{f} = 0$ almost everywhere on $B_{M-j}(0)$. Since j is an arbitrary positive integer, \hat{f} is equivalent to zero on \mathbb{Q}_p . \square

Another sufficient condition for axiom (c) was given in [9]:

Theorem 2.3. *If $\phi \in L^2(\mathbb{Q}_p)$ and the system $\{\phi(x - a) : a \in I_p\}$ is orthonormal, then axiom (c) of Definition 2.1 holds for the spaces V_j defined by (2.1).*

Theorem 2.4. *Let $\phi \in L^2(\mathbb{Q}_p)$, the spaces V_j , $j \in \mathbb{Z}$, be defined by (2.1), and let $\phi(\cdot - b) \in \cup_{j \in \mathbb{Z}} V_j$ for any $b \in \mathbb{Q}_p$. Axiom (b) of Definition 2.1 holds for the spaces V_j , $j \in \mathbb{Z}$, if and only if*

$$(2.5) \quad \bigcup_{j \in \mathbb{Z}} \text{supp } \hat{\phi}(p^j \cdot) = \mathbb{Q}_p.$$

¹Usually the terms “refinable function” and “scaling function” are synonyms in the literature, and they are used in both senses: as a solution to the refinement equation and as a function generating MRA. We separate here the meanings of these terms.

Remark 2.5. It is not difficult to see that the assumption $\phi(\cdot - b) \in \cup_{j \in \mathbb{Z}} V_j$ for any $b \in \mathbb{Q}_p$ is fulfilled whenever ϕ is a refinable function and $\widehat{\phi} \subset B_0(0)$. We will see that this assumption is also valid for a wide class of refinable functions ϕ for which $\widehat{\phi} \not\subset B_0(0)$.

Proof. First of all we show that the space $\overline{\cup_{j \in \mathbb{Z}} V_j}$ is invariant with respect to all shifts. Let $f \in \cup_{j \in \mathbb{Z}} V_j$, $b \in \mathbb{Q}_p$. Evidently, $\phi(p^{-k} \cdot - t) \in \cup_{j \in \mathbb{Z}} V_j$ for any $t \in \mathbb{Q}_p$ and for any $k \in \mathbb{Z}$. Since the L_2 -norm is invariant with respect to the shifts, it follows that $f(\cdot - b) \in \overline{\cup_{j \in \mathbb{Z}} V_j}$. If now $g \in \overline{\cup_{j \in \mathbb{Z}} V_j}$, then approximating g by the functions $f \in \cup_{j \in \mathbb{Z}} V_j$, again using the invariance of L_2 -norm with respect to the shifts, we derive $g(\cdot - b) \in \overline{\cup_{j \in \mathbb{Z}} V_j}$.

For $X \subset L^2(\mathbb{Q}_p)$, set $\widehat{X} = \{ \widehat{f} : f \in X \}$. By the Wiener theorem for L_2 (see, e.g., [10]; all the arguments of the proof given there may be repeated word for word with replacing \mathbb{R} by \mathbb{Q}_p), a closed subspace X of the space $L^2(\mathbb{Q}_p)$ is invariant with respect to the shifts if and only if $\widehat{X} = L_2(\Omega)$ for some set $\Omega \subset \mathbb{Q}_p$. If now $X = \overline{\cup_{j \in \mathbb{Z}} V_j}$, then $\widehat{X} = L_2(\Omega)$. Thus $X = L^2(\mathbb{Q}_p)$ if and only if $\Omega = \mathbb{Q}_p$. Set $\phi_j = \phi(p^{-j} \cdot)$, $\Omega_0 = \cup_{j \in \mathbb{Z}} \text{supp } \widehat{\phi}_j$ and prove that $\Omega = \Omega_0$. Since $\phi_j \in V_j$, $j \in \mathbb{Z}$, we have $\text{supp } \widehat{\phi}_j \subset \Omega$, and hence $\Omega_0 \subset \Omega$. Now assume that $\Omega \setminus \Omega_0$ contains a set of positive measure Ω_1 . Let $f \in V_j$. Given $\epsilon > 0$, there exists a function $f_\epsilon := \sum_{a \in I_p} \alpha_a \phi(p^j \cdot - a)$, where the sum is finite, such that $\|f - f_\epsilon\| < \epsilon$. Using (1.3), we see that $\text{supp } \widehat{f}_\epsilon \subset \text{supp } \widehat{\phi}(p^{-j} \cdot)$, which yields that $\widehat{f}_\epsilon(\xi) = 0$ for any $\xi \notin \Omega_1$. Due to the Plancherel theorem, it follows that $\widehat{f} = 0$ almost everywhere on Ω_1 . Hence the same is true for any $f \in \cup_{j \in \mathbb{Z}} V_j$. Passing to the limit we deduce that the Fourier transform of any $f \in X$ is equal to zero almost everywhere on Ω_1 , i.e., $L_2(\Omega) = L_2(\Omega_0)$. It remains to note that $\text{supp } \widehat{\phi}_j = \text{supp } \widehat{\phi}(p^j \cdot)$ \square

A real analog of Theorem 2.4 was proved by C. de Boor, R. DeVore and A. Ron in [13].

3. REFINABLE FUNCTIONS

We are going to study p -adic refinable functions ϕ . Let us restrict ourselves to the consideration of $\phi \in \mathcal{D}$. Evidently, each $\phi \in \mathcal{D}$ is a p^M -periodic function for some $M \in \mathbb{Z}$. Denote by \mathcal{D}_N^M the set of all p^M -periodic functions supported on $B_N(0)$. Taking the Fourier transform of the equality $\phi(x - p^M) = \phi(x)$, we obtain $\chi_p(p^M \xi) \widehat{\phi}(\xi) = \widehat{\phi}(\xi)$, which holds for all ξ if and only if $\text{supp } \widehat{\phi} \subset B_M(0)$. Thus, the set \mathcal{D}_N^M consists of all locally constant functions ϕ such that $\text{supp } \phi \subset B_N(0)$, $\text{supp } \widehat{\phi} \subset B_M(0)$.

Proposition 3.1. *Let $\phi, \psi \in L^2(\mathbb{Q}_p)$, $\text{supp } \phi, \text{supp } \psi \subset B_N(0)$, $N \geq 0$, and let $b \in I_p$, $|b|_p \leq p^N$. If*

$$(3.1) \quad \psi(\cdot - b) \in \overline{\text{span} \{ \phi(p^{-1} \cdot - a), a \in I_p \}}$$

then

$$(3.2) \quad \psi(x - b) = \sum_{k=0}^{p^{N+1}-1} h_{k,b}^\psi \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right) \quad \forall x \in \mathbb{Q}_p.$$

Proof. Given $\epsilon > 0$, there exist functions

$$f_\epsilon := \sum_{\substack{a \in I_p \\ |a|_p \leq p^{N+1}}} \alpha_a \phi(p^j \cdot -a), \quad g_\epsilon := \sum_{\substack{a \in I_p \\ |a|_p > p^{N+1}}} \alpha_a \phi(p^j \cdot -a),$$

where the sums are finite, such that $\|\psi(\cdot - b) - f_\epsilon - g_\epsilon\| < \epsilon$. If $x \in B_N(0)$, $|a|_p > p^{N+1}$, then $|p^{-1}x - a|_p > p^{N+1}$ and hence $\phi(p^{-1}x - a) = 0$. So, $g_\epsilon(x) = 0$ whenever $x \in B_N(0)$. If $x \notin B_N(0)$, then $\phi(x - b) = 0$ and $\phi(p^{-1}x - a) = 0$ for all $a \in I_p$, $|a|_p \leq p^{N+1}$. So, $\phi(\cdot - b) - f_\epsilon(x) = 0$ whenever $x \notin B_N(0)$. It follows that

$$\|\psi(\cdot - b) - f_\epsilon\|^2 = \int_{B_N(0)} |\psi(\cdot - b) - f_\epsilon|^2 = \int_{B_N(0)} |\psi(\cdot - b) - f_\epsilon - g_\epsilon|^2 \leq \epsilon^2.$$

Hence

$$\psi(\cdot - b) \in \overline{\text{span}\{\phi(p^{-1} \cdot -a), a \in I_p, |a|_p \leq p^{N+1}\}},$$

which implies (3.2). \square

Corollary 3.2. *If $\phi \in L^2(\mathbb{Q}_p)$ is a refinable function and $\text{supp } \phi \subset B_N(0)$, $N \geq 0$, then its refinement equation is*

$$(3.3) \quad \phi(x) = \sum_{k=0}^{p^{N+1}-1} h_k \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right) \quad \forall x \in \mathbb{Q}_p.$$

The proof immediately follows from Proposition 3.1.

Corollary 3.3. *Let $\phi \in L^2(\mathbb{Q}_p)$ be a scaling function of a MRA. If $\text{supp } \phi \subset B_N(0)$, $N \geq 0$, then ϕ is a refinable function satisfying (3.3).*

The proof follows by combining axiom (a) of Definition 2.1 with Proposition 3.1.

Taking the Fourier transform of (3.3) and using (1.3), we can rewrite the refinable equation in the form

$$(3.4) \quad \widehat{\phi}(\xi) = m_0\left(\frac{\xi}{p^N}\right) \widehat{\phi}(p\xi),$$

where

$$(3.5) \quad m_0(\xi) = \frac{1}{p} \sum_{k=0}^{p^{N+1}-1} h_k \chi_p(k\xi)$$

is a trigonometric polynomial. It is clear that $m_0(0) = 1$ whenever $\widehat{\phi}(0) \neq 0$.

Proposition 3.4. *If $\phi \in L^2(\mathbb{Q}_p)$ is a solution of refinable equation (3.3), $\widehat{\phi}(0) \neq 0$, $\widehat{\phi}(\xi)$ is continuous at the point 0, then*

$$(3.6) \quad \widehat{\phi}(\xi) = \widehat{\phi}(0) \prod_{j=0}^{\infty} m_0\left(\frac{\xi}{p^{N-j}}\right).$$

Proof. Since (3.3) implies (3.4), after iterating (3.4) J times, $J \geq 1$, we have

$$\widehat{\phi}(\xi) = \prod_{j=0}^J m_0\left(\frac{\xi}{p^{N-j}}\right) \widehat{\phi}(p^J \xi).$$

Taking into account that $\widehat{\phi}(\xi)$ is continuous at the point 0 and the fact that $|p^N \xi|_p = p^{-N} |\xi|_p \rightarrow 0$ as $N \rightarrow +\infty$ for any $\xi \in \mathbb{Q}_p$, we obtain (3.6). \square

Corollary 3.5. *If $\phi \in \mathcal{D}_N^M$ is a refinable function, $N \geq 0$, and $\widehat{\phi}(0) \neq 0$, then (3.6) holds.*

This statement follows immediately from Corollary 3.3 and Proposition 3.4.

Lemma 3.6. *Let $\widehat{\phi}(\xi) = C \prod_{j=0}^{\infty} m_0\left(\frac{\xi}{p^{N-j}}\right)$, where m_0 is a trigonometric polynomial with $m_0(0) = 1$ and $C \in \mathbb{R}$. If $\text{supp } \widehat{\phi} \subset B_M(0)$, then there exist at least $\left(p^{M+N} - \frac{\deg m_0}{p-1}\right)$ integers n such that $0 \leq n < p^{M+N}$ and $\widehat{\phi}\left(\frac{n}{p^M}\right) = 0$.*

Proof. First of all we note that $\widehat{\phi}$ is a p^N -periodic function satisfying (3.4). Denote by O_p the set of positive integers not divisible by p . Since $\text{supp } \widehat{\phi} \subset B_M(0)$, we have $\widehat{\phi}\left(\frac{k}{p^{M+1}}\right) = 0$ for all $k \in O_p$. By the definition of $\widehat{\phi}$ the equality $\widehat{\phi}\left(\frac{k}{p^{M+1}}\right) = 0$ holds if and only if there exists $\nu = 1 - N, \dots, M+1$ such that $m_0\left(\frac{k}{p^{N+\nu}}\right) = 0$. Set

$$\sigma_\nu := \left\{ l \in O_p : l < p^{N+\nu}, m_0\left(\frac{l}{p^{N+\nu}}\right) = 0, m_0\left(\frac{l}{p^{N+\mu}}\right) \neq 0 \forall \mu = 1 - N, \dots, \nu - 1 \right\},$$

$v_\nu := \#\sigma_\nu$. Evidently, $\sigma_\nu \subset O'_p$ for all ν , where $O'_p = \{k \in O_p : k < p^{M+N+1}\}$, and $\sigma_{\nu'} \cap \sigma_\nu = \emptyset$ whenever $\nu' \neq \nu$. If $\widehat{\phi}\left(\frac{k}{p^{M+1}}\right) = 0$ for some $k \in O_p$, then there exist a unique $\nu = 1 - N, \dots, M+1$ and a unique $l \in \sigma_\nu$ such that $k \equiv l \pmod{p^{N+\nu}}$. Moreover, for any $l \in \sigma_\nu$ there are exactly $p^{M-\nu+1}$ integers $k \in O'_p$ (including l) satisfying the above comparison. It follows that

$$(3.7) \quad \sum_{\nu=1-N}^{M+1} p^{M-\nu+1} v_\nu = \#O'_p = p^{M+N}(p-1).$$

Now if $l \in \sigma_\nu$, $\nu \leq M$, then $\widehat{\phi}\left(\frac{p^\gamma k}{p^M}\right) = 0$ for all $\gamma = 0, 1, \dots, M - \nu$, $k = l + rp^{N+\nu}$, $r = 0, 1, \dots, p^{M-\nu-\gamma} - 1$, i.e., each $l \in \sigma_\nu$ generates at least $1 + p + \dots + p^{M-\nu}$ distinct positive integers $n < p^{M+N}$ for which $\widehat{\phi}\left(\frac{n}{p^M}\right) = 0$. Hence

$$\begin{aligned} v &:= \#\left\{ n : n = 0, 1, \dots, p^{M+N} - 1, \widehat{\phi}\left(\frac{n}{p^M}\right) = 0 \right\} \geq \\ &\sum_{\nu=1-N}^M (1 + p + \dots + p^{M-\nu}) v_\nu = \frac{1}{p-1} \sum_{\nu=1-N}^M (p^{M-\nu+1} - 1) v_\nu = \\ &\frac{1}{p-1} \sum_{\nu=1-N}^{M+1} (p^{M-\nu+1} - 1) v_\nu. \end{aligned}$$

Since $\sum_{\nu=1-N}^{M+1} v_\nu \leq \deg m_0$, by using (3.7), we obtain

$$v \geq \frac{1}{p-1} \left(\sum_{\nu=1-N}^{M+1} p^{M-\nu+1} v_\nu - \deg m_0 \right) \geq p^{M+N} - \frac{\deg m_0}{p-1}.$$

□

Theorem 3.7. *Let $\phi \in \mathcal{D}_N^M$, $N \geq 0$ and $\widehat{\phi}(0) \neq 0$. If*

$$(3.8) \quad \phi(\cdot - b) \in \overline{\text{span}\{\phi(p^{-1} \cdot -a), a \in I_p\}}$$

for all $b \in I_p$, $|b|_p \leq p^N$, then there exist at least $p^{M+N} - p^N$ integers l such that $0 \leq l < p^{M+N}$ and $\widehat{\phi}\left(\frac{l}{p^M}\right) = 0$.

Proof. Let $b \in I_p$, $|b|_p \leq p^N$. Because of Proposition 3.1, we can rewrite (3.8) in the form

$$\phi(x-b) = \sum_{k=0}^{p^{N+1}-1} h_{k,b} \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right) \quad \forall x \in \mathbb{Q}_p.$$

Taking the Fourier transform, we obtain

$$(3.9) \quad \widehat{\phi}(\xi) \chi_p(b\xi) = m_b\left(\frac{\xi}{p^N}\right) \widehat{\phi}(p\xi), \quad \forall \xi \in \mathbb{Q}_p,$$

where m_b is a trigonometric polynomial, $\deg m_b < p^{N+1}$. Combining (3.16) for $b = 0$ with (3.16) for arbitrary b , we obtain

$$\widehat{\phi}(p\xi) \left(m_0\left(\frac{\xi}{p^N}\right) \chi_p(b\xi) - m_b\left(\frac{\xi}{p^N}\right) \right) = 0 \quad \forall \xi \in \mathbb{Q}_p,$$

which is equivalent to

$$(3.10) \quad F(\xi) := \widehat{\phi}(p^{N+1}\xi) (m_0(\xi) \chi_p(p^N b\xi) - m_b(\xi)) = 0 \quad \forall \xi \in \mathbb{Q}_p.$$

Since $\text{supp } F \subset B_{M+N+1}(0)$ and F is a 1-periodic function, (3.17) holds if and only if $\widehat{\phi}\left(\frac{l}{p^{M+N+1}}\right) = 0$, $l = 0, 1, \dots, p^{M+N+1} - 1$.

First suppose that $\deg m_0 \geq p^N(p-1)$, i.e.,

$$m_0(\xi) = \sum_{k=0}^K h_k \chi_p(k\xi), \quad h_K \neq 0,$$

where $K = K_N p^N + K_{N-1} p^{N-1} + \dots + K_0$, $K_j \in D_p$, $j = 0, 1, \dots, N$, $K_N = p-1$ (indeed, if $K_N < p-1$, then $\deg m_0 = K \leq (p-2)p^N + (p-1)(1+p+\dots+p^{N-1}) = p^{N+1} - p^N - 1 < p^N(p-1)$). Set $b := p - p^{-N}K$. It is not difficult to see that $b \in I_p$, $|b|_p \leq p^N$ and $K + bp^N = p^{N+1}$. We see that the degree of the polynomial $t(\xi) := m_0(\xi) \chi_p(p^N b\xi) - m_b(\xi)$ is exactly p^{N+1} , and hence there exist at most p^{N+1} integers l such that $0 \leq l < p^{M+N+1}$, $t\left(\frac{l}{p^{M+N+1}}\right) = 0$. Thus,

$$\# \left\{ l : l = 0, 1, \dots, p^{M+N+1} - 1, \widehat{\phi}\left(\frac{l}{p^M}\right) = 0 \right\} \geq p^{M+N+1} - p^{N+1}.$$

Taking into account that $\widehat{\phi}$ is a p^N -periodic function, we obtain

$$(3.11) \quad \# \left\{ l : l = 0, 1, \dots, p^{M+N} - 1, \widehat{\phi}\left(\frac{l}{p^M}\right) = 0 \right\} \geq p^{M+N} - p^N.$$

It remains to note that (3.11) is also fulfilled whenever $\deg m_0 < p^N(p-1)$ because of Lemma 3.6 and Corollary 3.5. \square

Theorem 3.8. Let $\phi, \psi \in \mathcal{D}_N^M$, $N \geq 0$, $\widehat{\phi}(0) \neq 0$, and let

$$(3.12) \quad \psi(\cdot) \in \overline{\text{span}\{\phi(p^{-1} \cdot -a), a \in I_p\}}$$

If there exist at least $p^{M+N} - p^N$ integers l such that $0 \leq l < p^{M+N}$ and $\widehat{\phi}\left(\frac{l}{p^M}\right) = 0$, then

$$(3.13) \quad \psi(x - b) = \sum_{a \in I_p} \alpha_{a,b}^\psi \phi(p^{-1}x - a) \quad \forall b \in \mathbb{Q}_p,$$

where the sum is finite. In particular, if ϕ is a refinable function, then

$$(3.14) \quad \phi(x - b) = \sum_{a \in I_p} \alpha_{a,b} \phi(p^{-1}x - a) \quad \forall b \in \mathbb{Q}_p.$$

Proof. First we assume that $b \in \mathbb{Q}_p$, $|b|_p \leq p^N$, $b \neq 0$, and prove that

$$(3.15) \quad \psi(x - b) = \sum_{k=0}^{p^{N+1}-1} g_{k,b} \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right) \quad \forall x \in \mathbb{Q}_p.$$

Because of Proposition 3.1, we have (3.15) for $b = 0$. Taking the Fourier transform of (3.15), we obtain

$$(3.16) \quad \widehat{\psi}(\xi) \chi_p(b\xi) = n_b\left(\frac{\xi}{p^N}\right) \widehat{\phi}(p\xi), \quad \forall \xi \in \mathbb{Q}_p,$$

where n_b is a trigonometric polynomial, $\deg n_b < p^{N+1}$. Substituting (3.16) for $b = 0$, we reduce (3.16) for arbitrary b to

$$\widehat{\phi}(p\xi) \left(n_0\left(\frac{\xi}{p^N}\right) \chi_p(b\xi) - n_b\left(\frac{\xi}{p^N}\right) \right) = 0 \quad \forall \xi \in \mathbb{Q}_p,$$

which is equivalent to

$$(3.17) \quad F(\xi) := \widehat{\phi}(p^{N+1}\xi) (n_0(\xi) \chi_p(p^N b\xi) - n_b(\xi)) = 0 \quad \forall \xi \in \mathbb{Q}_p.$$

Since $\text{supp } F \subset B_{M+N+1}(0)$ and F is a 1-periodic function, (3.17) is equivalent to

$$F\left(\frac{l}{p^{M+N+1}}\right) = 0, \forall l = 0, 1, \dots, p^{M+N+1} - 1,$$

which holds if and only if

$$(3.18) \quad n_b\left(\frac{l}{p^{M+N+1}}\right) = n_0\left(\frac{l}{p^{M+N+1}}\right) \chi_p\left(\frac{bl}{p^{M+1}}\right),$$

for all $l = 0, 1, \dots, p^{M+N+1} - 1$ such that $\widehat{\phi}\left(\frac{l}{p^M}\right) \neq 0$. Because of p^M -periodicity of $\widehat{\phi}$, there exist at least $p(p^{M+N} - p^N)$ integers $l = 0, 1, \dots, p^{M+N+1} - 1$ such that $\widehat{\phi}\left(\frac{l}{p^M}\right) \neq 0$. So, we can find n_b by solving the linear system (3.18) with respect to the unknown coefficients of n_b . Taking the Fourier transform of (3.16), we obtain

$$(3.19) \quad \psi(x - b) = \sum_{k=0}^{p^{N+1}-1} g_{k,b} \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right) \quad \forall x \in \mathbb{Q}_p.$$

Next let $b \in \mathbb{Q}_p$, $|b|_p = p^{N+1}$, i.e., $b = b_{N+1}p^{N+1} + b'$, $b_{N+1} \in D_p$, $b_{N+1} \neq 0$, $|b'|_p \leq p^N$. Using (3.19) with $b = b'$, we have

$$\psi(x - b) = \sum_{k=0}^{p^{N+1}-1} g_{k,b'} \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}} - \frac{b_{N+1}}{p^{N+2}}\right) = \sum_{k=0}^{p^{N+1}-1} g_{k,b'} \phi\left(\frac{x}{p} - \frac{pk + b_{N+1}}{p^{N+2}}\right).$$

Taking into account that

$$pk + b_{N+1} \leq p(p^{N+1} - 1) + (p - 1) = p^{N+2} - 1,$$

we derive

$$\psi(x - b) = \sum_{k=0}^{p^{N+2}-1} g_{k,b} \phi\left(\frac{x}{p} - \frac{k}{p^{N+2}}\right) \quad \forall x \in \mathbb{Q}_p.$$

Similarly, we can prove by induction on n that

$$\psi(x - b) = \sum_{k=0}^{p^{N+n+1}-1} g_{k,b} \phi\left(\frac{x}{p} - \frac{k}{p^{N+n+1}}\right) \quad \forall x \in \mathbb{Q}_p,$$

whenever $b \in \mathbb{Q}_p$, $|b|_p = p^{N+n}$. \square

Theorem 3.9. *A function $\phi \in \mathcal{D}_N^M$, $N \geq 0$, with $\widehat{\phi}(0) \neq 0$ generates a MRA if and only if*

- (1) ϕ is refinable;
- (2) there exist at least $p^{M+N} - p^N$ integers l such that $0 \leq l < p^{M+N}$ and $\widehat{\phi}\left(\frac{l}{p^M}\right) = 0$.

Proof. If ϕ is a scaling function of a MRA, then (1) follows from Corollary 3.3 and (2) follows from (1) and Theorem 3.7.

Now let conditions (1), (2) be fulfilled. Define the spaces V_j , $j \in \mathbb{Z}$, by (2.1). Axioms (d) and (e), evidently, hold. Axiom (a) follows from Theorem 3.8. Axiom (b) follows from Theorems 3.8 and 2.4. Axiom (c) follows from Theorems 2.2. \square

Example 3.10. Let $p = 2$, $N = 2$, $M = 1$ ϕ be defined by (3.6), where $\widehat{\phi}(0) \neq 0$, m_0 is given by (3.5), $m_0(1/4) = m_0(3/8) = m_0(7/16) = m_0(15/16) = 0$ and $m_0(0) = 1$. It is not difficult to see that $\text{supp } \widehat{\phi} \subset B_1(0)$, $\text{supp } \widehat{\phi} \not\subset B_0(0)$ and $\widehat{\phi}\left(\frac{1}{2}\right) = \widehat{\phi}\left(\frac{3}{2}\right) = \widehat{\phi}\left(\frac{5}{2}\right) = \widehat{\phi}(1) = 0$, i.e., all the assumptions of Theorem 3.9 are fulfilled.

Remark 3.11. The above example is typical. Similarly, taking into account the arguments of the proof of Lemma 3.6, one can easily construct a lot of functions ϕ generating a MRA for arbitrary p , $M > 0$ and large enough N . Moreover, it is possible to provide $\deg m_0 \leq 2^N$.

4. ORTHOGONAL SCALING FUNCTIONS

Now we are going to describe all orthogonal scaling functions $\phi \in \mathcal{D}_N^M$.

Theorem 4.1. *Let $\phi \in \mathcal{D}_N^M$, $M, N \geq 0$. If $\{\phi(x - a) : a \in I_p\}$ is an orthonormal system, then*

$$(4.1) \quad \sum_{l=0}^{p^{M+N}-1} \left| \widehat{\phi}\left(\frac{l}{p^M}\right) \right|^2 \chi_p\left(\frac{lk}{p^{M+N}}\right) = p^N \delta_{k0}, \quad k = 0, 1, \dots, p^N - 1.$$

Proof. Let $a \in I_p$. Due to the orthonormality of $\{\phi(x - a) : a \in I_p\}$, using the Plancherel theorem, we have

$$\delta_{a0} = \langle \phi(\cdot), \phi(\cdot - a) \rangle = \int_{\mathbb{Q}_p} \phi(x) \overline{\phi(x - a)} dx = \int_{B_M(0)} |\widehat{\phi}(\xi)|^2 \chi_p(a\xi) d\xi.$$

Let $\xi \in B_M(0)$. There exists a unique $l = 0, 1, \dots, p^{M+N}-1$ such that $\xi \in B_{-N}(b_l)$, $b_l = \frac{l}{p^M}$. It follows that

$$\begin{aligned} \int_{B_M(0)} |\widehat{\phi}(\xi)|^2 \chi_p(a\xi) d\xi &= \sum_{k=0}^{p^{M+N}-1} \int_{|\xi-b_l|_p \leq p^{-N}} |\widehat{\phi}(\xi)|^2 \chi_p(a\xi) d\xi \\ &= \sum_{l=0}^{p^{M+N}-1} |\widehat{\phi}(b_l)|^2 \int_{|\xi-b_l|_p \leq p^{-N}} \chi_p(a\xi) d\xi = \sum_{l=0}^{p^{M+N}-1} |\widehat{\phi}(b_l)|^2 \chi_p(ab_l) \int_{|\xi|_p \leq p^{-N}} \chi_p(a\xi) d\xi \\ &= \frac{1}{p^N} \Omega(|p^N a|_p) \sum_{l=0}^{p^{M+N}-1} |\widehat{\phi}(b_l)|^2 \chi_p(ab_l). \end{aligned}$$

To prove (4.1) it only remains to note that $\Omega(|p^N a|_p) = 0$ whenever $a \in I_p$, $p^N a \neq 0, 1, \dots, p^N - 1$. \square

Lemma 4.2. *Let c_0, \dots, c_{n-1} be mutually distinct elements of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Suppose that there exist nonzero reals x_j , $j = 0, 1, \dots, n-1$, such that*

$$(4.2) \quad \sum_{j=0}^{n-1} c_j^k x_j = \delta_{k0}, \quad k = 0, 1, \dots, n-1.$$

Then $x_j = 1/n$ for all j , and up to reordering

$$(4.3) \quad c_j = c_0 e^{2\pi i j/n}, \quad j = 0, 1, \dots, n-1.$$

Proof. In accordance with Cramer's rule we have $x_j = \frac{\Delta_j}{\Delta}$, $0 \leq j \leq n-1$, where $\Delta = V(c)$ is the Vandermonde determinant corresponding to $c = (c_0, \dots, c_{n-1})$, and Δ_j is obtained from Δ by replacing the j -th column with the transpose of the row $(1, 0, \dots, 0)$. A straightforward computation shows that

$$\Delta_j = (-1)^j V(c^{(j)}) \prod_{k \neq j} c_k,$$

where $c^{(j)}$ is obtained from c by removing the j -th coordinate. Thus,

$$(4.4) \quad \begin{aligned} x_j &= (-1)^j \frac{V(c^{(j)})}{V(c)} \prod_{k \neq j} c_k = (-1)^j \prod_{k \neq j} c_k \prod_{\substack{k > l \\ k, l \neq j}} (c_k - c_l) / \prod_{k > l} (c_k - c_l) \\ &= \prod_{k \neq j} \frac{c_k}{c_k - c_j} = \prod_{k \neq j} \frac{1}{1 - c_k^{-1} c_j}. \end{aligned}$$

Next, for any $\alpha \in \mathbb{R}$, we have

$$1 - e^{i\alpha} = 2 \sin \frac{\alpha}{2} \left(\sin \frac{\alpha}{2} - i \cos \frac{\alpha}{2} \right) = 2 \sin \frac{\alpha}{2} e^{i(\frac{\alpha}{2} - \frac{\pi}{2})}.$$

Let us define α_j , $j = 0, 1, \dots, n-1$, by $c_j = e^{i\alpha_j}$. Then from the above arguments and (4.4) it follows that

$$x_j = \prod_{k \neq j} \frac{1}{1 - c_k^{-1} c_j} = e^{i\gamma} \sum_{k \neq j} \left(2 \sin \frac{\alpha_k - \alpha_j}{2} \right)^{-1},$$

where

$$\gamma = \sum_{k \neq j} \frac{\alpha_k - \alpha_j + \pi}{2} = \theta - \frac{n}{2} \alpha_j, \quad \theta = \frac{1}{2} \left((n-1)\pi + \sum_{k=0}^{n-1} \alpha_k \right)$$

By the lemma's hypothesis $x_j \in \mathbb{R}$, whence $\gamma \equiv 0 \pmod{\pi}$ and consequently $n\alpha_j \equiv 2\theta \pmod{2\pi}$. Thus up to reordering $\alpha_j = \alpha_0 + \frac{2\pi j}{n}$, which implies (4.3), and consequently that $x_j = 1/n$ for all j . \square

Theorem 4.3. *Let $\phi \in \mathcal{D}_N^M$ be an orthogonal scaling function and $\widehat{\phi}(0) \neq 0$. Then $\text{supp } \widehat{\phi} \subset B_0(0)$.*

Proof. Without loss of generality, we can assume that $M, N \geq 0$. Combining Theorems 3.9 and 4.1, we have

$$\sum_{j=0}^{p^N-1} \left| \widehat{\phi} \left(\frac{l_j}{p^M} \right) \right|^2 \chi_p \left(\frac{l_j k}{p^{M+N}} \right) = p^N \delta_{k0}, \quad k = 0, 1, \dots, p^N - 1.$$

By Lemma 4.2, $l_j = l_0 + jp^M$ and $\widehat{\phi} \left(\frac{l_j}{p^M} \right) = 1$. Taking into account that $\widehat{\phi}(0) \neq 0$, we deduce $l_0 = 0$, i.e., $\widehat{\phi}(j) = 1$, $j = 0, 1, \dots, p^N - 1$. Since $\widehat{\phi}$ is a p^N -periodic function, it follows from Theorem 3.9 that $\widehat{\phi} \left(\frac{l}{p^M} \right) = 0$ for all $l \in \mathbb{Z}$ not divisible by p^M . This yields $\text{supp } \widehat{\phi} \subset B_0(0)$. \square

So any test function ϕ generating a MRA belongs to the class \mathcal{D}_N^0 . All such functions were described in [9]. The following theorem summarizes these results.

Theorem 4.4. *Let $\widehat{\phi}$ be defined by (3.6), where m_0 is the trigonometric polynomial (3.5) with $m_0(0) = 1$. If $m_0 \left(\frac{k}{p^{N+1}} \right) = 0$ for all $k = 1, \dots, p^{N+1} - 1$ not divisible by p , then $\phi \in \mathcal{D}_N^0$. If, furthermore, $|m_0 \left(\frac{k}{p^{N+1}} \right)| = 1$ for all $k = 1, \dots, p^{N+1} - 1$ divisible by p , then $\{\phi(x-a) : a \in I_p\}$ is an orthonormal system. Conversely, if $\text{supp } \widehat{\phi} \subset B_0(0)$ and the system $\{\phi(x-a) : a \in I_p\}$ is orthonormal, then $|m_0 \left(\frac{k}{p^{N+1}} \right)| = 0$ whenever k is not divisible by p , and $|m_0 \left(\frac{k}{p^{N+1}} \right)| = 1$ whenever k is divisible by p , $k = 1, 2, \dots, p^{N+1} - 1$.*

5. CONSTRUCTION OF WAVELET FRAMES

Definition 5.1. Let H be a Hilbert space. A system $\{f_n\}_{n=1}^\infty \subset H$ is said to be a frame if there exist positive constants A, B (frame boundaries) such that

$$A\|f\|^2 \leq \sum_{n=1}^\infty |\langle f, f_n \rangle|^2 \leq B\|f\|^2 \quad \forall f \in H.$$

We are interested in the construction of p -adic wavelet frames, i.e., frames in $L_2(\mathbb{Q}_p)$ consisting of functions $p^{j/2} \psi^{(\nu)}(p^{-j} \cdot -a)$, $a \in I_p$, $\nu \in A$, where A is a finite set.

We will restrict ourselves to the consideration of the case $p = 2$.

Our general scheme of construction looks as follows. Let $\{V_j\}_{j \in \mathbb{Z}}$ be a MRA. As above, we define the wavelet space W_j , $j \in \mathbb{Z}$, as the orthogonal complement of V_j in V_{j+1} , i.e., $V_{j+1} = V_j \oplus W_j$. It is not difficult to see that $f \in W_j$ if and

only if $f(2^j \cdot) \in W_0$, and $W_j \perp W_k$ whenever $j \neq k$. If now there exists a function $\psi \in L_2(Q_2)$ (wavelet function) such that

$$(5.1) \quad W_0 = \overline{\text{span} \{ \psi(\cdot - a), a \in I_2 \}},$$

then we have a wavelet system $\{2^{j/2}\psi(2^{-j} \cdot - a), a \in I_2, j \in \mathbb{Z}\}$. It will be proved that such a system is a frame in $L_2(Q_2)$ whenever ψ is compactly supported.

Theorem 5.2. *Let $\{V_j\}_{j \in \mathbb{Z}}$ be a MRA, ψ be a wavelet function. If ψ is compactly supported, then the corresponding wavelet system $\{2^{j/2}\psi(2^{-j} \cdot - a), a \in I_2, j \in \mathbb{Z}\}$ is a frame in $L_2(Q_2)$.*

Proof. First we will prove that the system $\{\psi(\cdot - a), a \in I_2\}$ is a frame in the wavelet space W_0 . Let $\text{supp } \psi \subset B_N(0)$, $N \geq 0$. Set

$$\begin{aligned} W_0^0 &= \text{span} \{ \psi(\cdot - a), a \in I_2, |a|_2 \leq 2^N \}, \\ W_0^n &= \text{span} \{ \psi(\cdot - a), a \in I_2, |a|_2 = 2^{N+n} \}, \quad n \in \mathbb{N}. \end{aligned}$$

It is not difficult to see that the spaces W_0^n , $n = 0, 1, \dots$, are mutually orthogonal. Each function $f \in W_0$ may be represented in the form $f = f^0 + f^1 + \dots$, where $f^0 = f|_{B_N(0)}$, $f^n = f|_{B_{N+n}(0) \setminus B_{N+n-1}(0)}$, $n \in \mathbb{N}$. Due to (5.1), given $\epsilon > 0$, there exists a finite sum $\sum_{a \in I_2} \alpha_a \psi(\cdot - a) =: f_\epsilon$ such that $\|f - f_\epsilon\| < \epsilon$. If $|x|_2 \leq 2^N$, then $f_\epsilon(x) = \sum_{\substack{a \in I_2 \\ |a|_2 \leq 2^N}} \alpha_a \psi(x - a) =: f_\epsilon^0(x)$. Since $\text{supp } f^0 \subset B_N(0)$, $\text{supp } f_\epsilon^0 \subset B_N(0)$, we have

$$\|f - f_\epsilon\|^2 \geq \int_{B_N(0)} |f - f_\epsilon|^2 = \int_{B_N(0)} |f^0 - f_\epsilon^0|^2 = \|f^0 - f_\epsilon^0\|^2.$$

It follows that $f^0 \in W_0^0$. Similarly, $f^n \in W_0^n$, $n \in \mathbb{N}$. Thus we proved that

$$(5.2) \quad W_0 = W_0^0 \oplus W_0^1 \oplus W_0^2 \oplus \dots$$

Since W_0^0 is a finite dimensional space and $\{\psi(\cdot - a), a \in I_2, |a|_2 \leq 2^N\}$ is a representing system for W_0^0 , this system is a frame. Hence there exist positive constants A, B such that

$$A\|f^0\|^2 \leq \sum_{a \in I_2} |\langle f^0, \psi(\cdot - a) \rangle|^2 \leq B\|f^0\|^2 \quad \forall f \in W_0^0.$$

If $f^1 \in W_0^1$, we have

$$\begin{aligned} \sum_{\substack{a \in I_2 \\ |a|_2 = 2^{N+1}}} |\langle f^1, \psi(\cdot - a) \rangle|^2 &= \sum_{\substack{a \in I_2 \\ |a|_2 \leq 2^N}} |\langle f^1, \psi(\cdot - a - 2^{-N-1}) \rangle|^2 = \\ \sum_{\substack{a \in I_2 \\ |a|_2 \leq 2^N}} |\langle f^1(\cdot + 2^{-N-1}), \psi(\cdot - a) \rangle|^2 &\geq A\|f^1(\cdot + 2^{-N-1})\|^2 = A\|f^1\|^2. \end{aligned}$$

Let now $f^n \in W_0^n$, $n > 1$. Then

$$\begin{aligned}
\sum_{\substack{a \in I_2 \\ |a|_2 = 2^{N+n}}} |\langle f^n, \psi(\cdot - a) \rangle|^2 &= \sum_{k=0}^{2^{n-1}-1} \sum_{\substack{a \in I_2 \\ |a|_2 \leq 2^N}} \left| \left\langle f^n, \psi \left(\cdot - a - \frac{2k+1}{2^{N+n}} \right) \right\rangle \right|^2 = \\
&\sum_{k=0}^{2^{n-1}-1} \sum_{\substack{a \in I_2 \\ |a|_2 \leq 2^N}} \left| \left\langle f^n \left(\cdot + \frac{2k+1}{2^{N+n}} \right) \Omega(|2^N \cdot|), \psi(\cdot - a) \right\rangle \right|^2 \geq \\
A \sum_{k=0}^{2^{n-1}-1} \left\| f^n \left(\cdot + \frac{2k+1}{2^{N+n}} \right) \Omega(|2^N \cdot|) \right\|^2 &= A \sum_{k=0}^{2^{n-1}-1} \left\| f^n \Omega \left(\left| 2^N \left(\cdot - \frac{2k+1}{2^{N+n}} \right) \right| \right) \right\|^2 = \\
&A \sum_{k=0}^{2^{n-1}-1} \left\| f^n \Big|_{B_N(\frac{2k+1}{2^{N+n}})} \right\|^2 = A \|f^n\|^2.
\end{aligned}$$

Taking into account (5.2), we derive

$$(5.3) \quad A \|f\|^2 \leq \sum_{a \in I_2} |\langle f, \psi(\cdot - a) \rangle|^2 \quad \forall f \in W_0.$$

Similarly we can prove the upper frame estimation

$$(5.4) \quad \sum_{a \in I_2} |\langle f, \psi(\cdot - a) \rangle|^2 \leq B \|f\|^2 \quad \forall f \in W_0.$$

Combining (5.3) with (5.4), we deduce that the system $\{\psi(\cdot - a), a \in I_2\}$ is a frame in W_0 . Evidently, the system $\{2^{j/2}\psi(2^{-j} \cdot - a), a \in I_2, j \in \mathbb{Z}\}$ is a frame in W_j with the same frame boundaries for any $j \in \mathbb{Z}$. Since $\bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{Q}_2)$, it follows that the union of these frames is a frame in $L_2(\mathbb{Q}_2)$. \square

Now we discuss how to construct a desirable wavelet function ψ . Let a MRA $\{V_j\}_{j \in \mathbb{Z}}$ is generated by a scaling function $\phi \in \mathcal{D}_N^M$. First of all we should provide $\psi \in V_1$. Let us look for ψ in the form

$$\psi(x) = \sum_{k=0}^{2^{N+1}-1} g_k \phi\left(\frac{x}{2} - \frac{k}{2^{N+1}}\right)$$

Taking the Fourier transform of (3.3) and using (1.3), we have

$$\widehat{\psi}(\xi) = n_0\left(\frac{\xi}{2^N}\right) \widehat{\phi}(2\xi),$$

where n_0 is a trigonometric polynomial (*wavelet mask*) given by

$$n_0(\xi) = \frac{1}{2} \sum_{k=0}^{2^{N+1}-1} g_k \chi_2(k\xi)$$

Evidently, $\psi \in \mathcal{D}_N^M$. By Theorem 3.7, there exist at least $2^{M+N} - 2^N$ integers l such that $0 \leq l < 2^{M+N}$, $\widehat{\phi}\left(\frac{l}{2^M}\right) = 0$. Choose n_0 satisfying the following property: if $\widehat{\phi}\left(\frac{l}{2^M}\right) \neq 0$ for some $l = 0, 1, \dots, 2^{M+N} - 1$, then $n_0\left(\frac{l}{2^{M+N}}\right) = 0$. This yields that $\widehat{\psi}\left(\frac{l}{2^M}\right) = 0$ whenever $\widehat{\phi}\left(\frac{l}{2^M}\right) \neq 0$, $0 \leq l < 2^{M+N}$.

Let $a, b \in I_2$. Using the Plancherel theorem and the arguments of Theorem 3.7, we have

$$\begin{aligned} \langle \phi(\cdot - a), \psi(\cdot - b) \rangle &= \int_{\mathbb{Q}_2} \phi(x - a) \overline{\psi(x - b)} dx = \int_{B_M(0)} \widehat{\phi}(\xi) \overline{\widehat{\psi}(\xi)} \chi_2((b - a)\xi) d\xi = \\ &= \sum_{k=0}^{2^{M+N}-1} \int_{|\xi - 2^{-M}l|_2 \leq 2^{-N}} \widehat{\phi}(\xi) \overline{\widehat{\psi}(\xi)} \chi_2((b - a)\xi) d\xi = \\ &= \sum_{l=0}^{2^{M+N}-1} \widehat{\phi}\left(\frac{l}{2^M}\right) \overline{\widehat{\psi}\left(\frac{l}{2^M}\right)} \int_{|\xi - 2^{-M}l|_2 \leq 2^{-N}} \chi_2(a\xi) d\xi = 0. \end{aligned}$$

It follows that $\overline{\text{span}\{\psi(\cdot - a), a \in I_2\}} \perp V_0$. On the other hand, due to Theorem 3.8, we have $\overline{\text{span}\{\psi(\cdot - a), a \in I_2\}} \subset V_1$. Hence,

$$(5.5) \quad \overline{\text{span}\{\psi(\cdot - a), a \in I_2\}} \subset W_0.$$

It is clear from that proof of Theorem 3.8 that

$$(5.6) \quad \psi\left(x - \frac{l}{p^N}\right) = \sum_{k=0}^{p^{N+1}-1} g_{kl} \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right), \quad l = 0, 1, \dots, 2^N - 1,$$

$$(5.7) \quad \phi\left(x - \frac{l}{p^N}\right) = \sum_{k=0}^{p^{N+1}-1} h_{kl} \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right), \quad l = 0, 1, \dots, 2^N - 1.$$

Consider these equalities as a linear system with respect to the unknowns $X_k := \phi\left(\frac{x}{p} - \frac{k}{p^{N+1}}\right)$, $k = 0, 1, \dots, 2^{N+1} - 1$. If the system (5.6), (5.7) has a solution, then

$$\text{span}\left\{\phi\left(\frac{\cdot}{p} - a\right), a \in I_2, |a|_2 \leq 2^{N+1}\right\} \subset \text{span}\{\psi(\cdot - a), a \in I_2, |a|_2 \leq 2^N\}.$$

This evidently implies $W_0 \subset \overline{\text{span}\{\psi(\cdot - a), a \in I_2\}}$. Taking into account (5.5), we deduce that ψ is a wavelet function.

It is not quite clear whether the system (5.6), (5.7) has a solution for arbitrary ϕ and ψ , but we will show how to succeed in the case $\deg m_0 \leq 2^N$. The construction of such masks can easily be done (see Example 3.10 and Remark 3.11).

Assume that $\deg m_0 \leq 2^N$. In this case

$$(5.8) \quad \widehat{\phi}(\xi) \chi_2\left(\frac{l\xi}{2^N}\right) = m_{l/2^N}\left(\frac{\xi}{2^N}\right) \widehat{\phi}(2\xi), \quad l = 0, 1, \dots, 2^N - 1,$$

where $m_{l/2^N}(\xi) = \chi_2(l\xi) m_0(\xi)$, $\deg m_{l/2^N} < 2^{N+1}$. It is clear that a wavelet mask n_0 can also be chosen in such a way that $\deg n_0 \leq 2^N$, and we have

$$(5.9) \quad \widehat{\psi}(\xi) \chi_2\left(\frac{l\xi}{2^N}\right) = n_{l/2^N}\left(\frac{\xi}{2^N}\right) \widehat{\psi}(2\xi), \quad l = 0, 1, \dots, 2^N - 1,$$

where $n_{l/2^N}(\xi) = \chi_2(l\xi) n_0(\xi)$, $\deg n_{l/2^N} < 2^{N+1}$. Taking the Fourier transform of (5.8), (5.9), we see that the matrix of the system (5.6), (5.7) looks as follows:

$$\begin{pmatrix} g_0 & g_1 & \cdots & g_{2^N-1} & g_{2^N} & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & g_{2^N-2} & g_{2^N-1} & g_{2^N} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_0 & g_1 & g_2 & \cdots & g_{2^N} \\ h_0 & h_1 & \cdots & h_{2^N-1} & h_{2^N} & 0 & \cdots & 0 \\ 0 & h_0 & \cdots & h_{2^N-2} & h_{2^N-1} & h_{2^N} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & h_0 & h_1 & h_2 & \cdots & h_{2^N} \end{pmatrix}$$

The determinant of this matrix is so called resultant. The resultant is not equal to zero if and only if the algebraic polynomials with the coefficients g_0, g_1, \dots, g_{2^N} and h_0, h_1, \dots, h_{2^N} respectively do not have joint zeros (see, e.g., [14]). But this holds because the trigonometric polynomials m_0 and n_0 do not have joint zeros by construction (taking care of not adding extra zeros).

REFERENCES

1. Mallat, S. (1988). Multiresolution representation and wavelets, Ph. D. Thesis, University of Pennsylvania, Philadelphia, PA.
2. Meyer, Y. (D cembre 1986). Ondelettes et fonctions splines, S minaire EDP. Paris.
3. S.V. Kozyrev, Wavelet analysis as a p -adic spectral analysis, Izvestia Akademii Nauk, Seria Math. **66** no. 2 (2002) 149–158.
4. Daubechies I. *Ten Lectures on wavelets*, CBMS-NSR Series in Appl. Math., SIAM, 1992.
5. S.V. Kozyrev, p -Adic pseudodifferential operators and p -adic wavelets, Theor. Math. Physics **138**, no. 3 (2004) 1–42.
6. J.J. Benedetto, and R.L. Benedetto, A wavelet theory for local fields and related groups, The Journal of Geometric Analysis **3** (2004) 423–456.
7. A.Yu. Khrennikov, V.M. Shelkovich, p -Adic multidimensional wavelets and their application to p -adic pseudo-differential operators, (2006). <http://arxiv.org/abs/math-ph/0612049>
8. V. M. Shelkovich, M. Skopina p -Adic Haar multiresolution analysis and pseudo-differential operators. <http://arxiv.org/abs/0705.2294>
9. A.Yu. Khrennikov, V.M. Shelkovich, M. Skopina, p -Adic refinable functions and MRA-based wavelets <http://arxiv.org/abs/0711.2820>
10. I. Novikov, V. Protassov, and M. Skopina, Wavelet Theory (in Russian). Moscow: Fizmatlit, 2005.
11. Rudin W. Real and complex analysis, New York: McGraw-Hill, 1974.
12. V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, p -Adic analysis and mathematical physics. World Scientific, Singapore, 1994.
13. de Boor C., DeVore R., Ron A. On construction of multivariate (pre) wavelets // Constr. Approx. 1993. V. 9. P. 123-166.
14. S. Lang, Algebra. Graduate texts in mathematics 211, Springer-Verlag, Berlin, 2002.

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